The EWMA Run Length Distribution Analysis Tool (ERLDAT)

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Abstract

The run length distribution of a control chart completely characterizes the performance of a control chart. Quality practitioners usually do not have access to the run length distribution but rely on the average run length (ARL) to design and evaluate the performance of an exponentially weighted moving average (EWMA) control charts. This paper will present a web-based tool that provides users access to the run length distribution for a EWMA control chart with known parameters. The web-based tool will calculate the run length distribution, percentiles of the run length distribution as well as the mean (ARL) and variance (VRL) of the run length distribution. Additional functionality of the web-based tool includes plotting the run length distribution functions, building tables of the quantiles of the run length distribution, finding the smoothing parameter \( \lambda \) for an EWMA control chart for fixed control limit that satisfies ARL, VRL or percentile performance, and finding the control chart limit \( k \) for an EWMA control chart that satisfies ARL, VRL or percentile performance. This tools and these techniques will enable quality practitioners to better design and evaluate EWMA control charts.

Keywords

Exponentially weighted moving average (EWMA) control chart, run length distribution, average run length (ARL).

1. Introduction

The exponentially weighted moving average (EWMA) control chart is a popular and commonly used control chart. One reason for the popularity of the EWMA control chart is its ability to detect small shifts in the mean. The EWMA statistic for observation \( t \) is defined to be

\[
Q_t = (1 - \lambda) Q_{t-1} + \lambda X_t
\]

where the smoothing constant is \( \lambda (0 < \lambda \leq 1) \) and the initial value of the EWMA statistic is \( Q_0 \) which is assigned a value between the lower and upper control limits. The control limits for the EWMA control chart are based upon the steady-state variance of the EWMA statistic and are defined to be

\[
UCL = \mu + k \left( \frac{\sigma}{\sqrt{n}} \right) \sqrt{\frac{\lambda}{2 - \lambda}}
\]
\[
LCL = \mu - k \left( \frac{\sigma}{\sqrt{n}} \right) \sqrt{\frac{\lambda}{2 - \lambda}}
\]

where \( \mu = E(X_t) \) and \( \sigma^2 = V(X_t) \). It is commonly assumed that the observations, \( x_t \), are observations from an independent normal distribution. The process of designing an EWMA control chart involves selecting values of \( \lambda \) and \( k \) that provide satisfactory control chart performance. The performance of control charts is defined by its run length distribution. Often the run length distribution is not know but the mean of the run length distribution, the average run length (ARL) is used as the control chart performance criterion. Two cases of the ARL are used to evaluate control chart performance. The first case is the on-target ARL in which no shift in the mean has occurred. Ideally, the on-target ARL will be large because when there is no shift in the mean a control chart signal is a false alarm. The second ARL case occurs when a shift in the mean of the process has occurred and is called the off-target ARL. Ideally the off-target
ARL will be small. Less frequently percentiles of the run length distribution, such as the median, or the variance of the run length (VRL) distribution are used as performance criteria. Unfortunately, quality practitioners often do not have performance information to make informed EWMA control chart design decisions.

This paper presents the EWMA Run Length Distribution Analysis Tool (ERLDAT) that provides quality practitioners a web-based application to design and analyze the performance of EWMA control charts for standard normal data with subgroups of size 1 \( (n = 1) \). The second section of this paper covers the background information on the performance analysis and the design of EWMA control charts. The third section provides an explanation of the methodology used to develop ERLDAT. Next, the paper presents several performance evaluations and examples cases from the literature for comparison discussion. And finally, an invitation to use the tools with concluding remarks, and future research vision and direction is provided.

2. Background

The background section provides a review of previous techniques for evaluating the performance of EWMA control charts and techniques for designing EWMA control charts.

2.1 Performance Analysis of the EWMA Control Chart

In the initial stages of statistical computing, Crowder [1] provided a FORTRAN program to calculate the ARL of the EWMA control chart using an integral equation formulation. In this program, users were required to specify (hard-code in) the values of \( \lambda \) and \( k \) and the program would produce a table of ARL results for varying shift sizes. Crowder [2] also presented tabulated results based upon integral calculations of the mean and standard deviation of the run length distribution for limited cases in his 1987 Technometrics paper. Later on, Saccucci and Lucas [3] presented a FORTRAN program that calculated the average run length of EWMA and combined EWMA-Shewhart charts using the Markov chain approach. Gan [4] also provided a FORTRAN program to calculate percentage points of the run length distribution of the EWMA control chart using an integral equation formulation. And Waldmann [5] provided bounds for the run length distribution of the EWMA control chart using integral equations that do not require the calculation of the entire run length distribution of the EWMA control chart.

The ERLDAT tool presented here is a method that is computationally superior to previous methods of computing the run length distribution because it does not require the calculation of the entire run length distribution. Rather it was observed that the survival function of the run length distribution converges to defined constants often very quickly. These constants can then be used to calculate percentage points of the run length distribution, the average run length and variance of the run length distribution.

2.2 Design of EWMA Control Charts

The process of designing an EWMA control chart involves selecting values of \( \lambda \) and \( k \) that provide satisfactory control chart performance. Montgomery [6] provides guidelines for selecting values of \( \lambda \) that are in the range \( 0.05 \leq \lambda \leq 0.25 \) and he observed that these values of \( \lambda \) work well with \( k = 3 \). Montgomery’s text also provides a table from Lucas and Saccucci [7] for designing EWMA control charts with an on-target ARL of 500. Going a step further, Crowder [8] provided a method of designing optimal EWMA control charts based upon a graphical method. He proposes a methodology in which first the on-target ARL is first selected. The second step is to specify the size of the shift in the mean that you want the control chart to optimally detect. Using figures, the value of \( \lambda \) is found that will optimally detect the specified shift with the specified on-target ARL. The final step is to find the value of \( k \) by using a second figure containing the value of \( \lambda \) and the on-target ARL. Crowder recommended that a sensitivity analysis be performed by comparing multiple pairs of \( (\lambda, k) \). Furthermore, Gan [9] provided a graphical method similar to the one given by Crowder to design EWMA control charts based upon the median run length instead of the average run length.
3. Methodology

ERLDAT is a JavaServer Faces implementation of the Waldmann [5] bound on the run length distribution of the EWMA control chart that facilitates quality practitioners to design and analyze EWMA control charts. JavaServer Faces (JSF) [10] is a “java specification for building component-based user interfaces for web applications.” JSF utilizes Facelets as a default templating system and is part of the Java Platform, Enterprise Edition. The run length calculations are implemented as methods within a JavaBean. The run length distribution is formulated as integral equations and solved numerically using 64-point Gaussian quadrature. A beta version of the ERLDAT application is available at [http://quality.engr.utpa.edu/Erldat](http://quality.engr.utpa.edu/Erldat). All screen captures in this draft are taken from the development server and will display a different url than the url of the production server. The entry screen of the ERLDAT application is shown in Figure 1. Notice that two functions are provided for users: Design an EWMA control chart and Evaluate an EWMA Control Chart. Users must first design an EWMA control chart before analyzing an EWMA control chart.

Several packages were used in the creation of the ERLDAT web application. Much of the user interface including the data tables, graphs and ability to export information to pdf and Excel files was provided by using the community (open source) version of PrimeFaces [11]. PrimeFaces also utilizes Apache POI [12] and iText [13]. The normal probability calculation were performed the JSci library [14].

![Figure 1: ERLDAT Entry Screen](image)

3.1 ERLDAT Design Page

The ERLDAT Design Page is shown in Figure 2. Users are provided seven different methods for designing an EWMA control chart. These seven methods can be divided into three categories. The first category is to complete specify an EWMA control chart. In this case, the user specifies the values for \( \lambda \), \( k \), \( Q_0 \) and the size of the shift in the mean (if any). Note that the shift size is specified in units of the standard deviation of the observations. Thus, a shift of size 1.0 is a 1.0\( \sigma \) increase in the mean of the process. The remaining six methods can be divided into two categories: find \( \lambda \) for specified values of \( k \), \( Q_0 \) and shift size to meet performance criteria or find \( k \) for specified values of \( \lambda \), \( Q_0 \), and shift size. The performance criterion available are ARL, VRL or quantile performance. In the method of designing for quantile performance the user must specified the desired quantile (e.g. for median specify 0.5) and the desired value of the quantile (e.g. \( \tilde{x} = 600 \)). The ARL, VRL and quantile performance searches utilize the bisection method.
3.2 ERLDAT Evaluation Page

The ERLDAT Evaluate page is shown in Figure 3. Figure 3 shows the results from the first design case from an example taken from Gan [4] that has an on-target ARL of 1000. The Evaluate Page contains a data table that displays the parameters of the control chart and the chart’s ARL and VRL performance. The data table can be exported to pdf or Excel files. Users can also evaluate quantiles of the run length distribution, the probability mass function of the run length distribution, the cumulative distribution function of the run length distribution and the survival/reliability function of the run length distributions. Examples of the results from the quantiles and various distribution functions will be provided in the Results Section.
4. Results

The Results section will present three case studies that illustrate the major design and analysis functionality of ERL-DAT. It is not possible in this paper to present all of the design and analysis capabilities of ERLDAT.

4.1 Case Study 1

The first case study focuses on finding the value of $\lambda$ for an EWMA control chart with parameters $k = 3.0$, $Q_0 = 0.0$, no shift in the mean, and with a target ARL that has a value of 750. This is a functionality that is not currently available in the literature. Quality practitioners could utilize a trial and error technique using a computer program such as the one provided by Gan [4]. The control chart parameters are entered using Design Case 2 shown in Figure 4. The Evaluate Page shown in Figure 5 shows that the value of $\lambda$ should be 0.1198, with an ARL of 750.0015 and VRL of 551283.035 as seen in the literature.

A graph of the quantile function is provided in Figure 6. If users right-click on a graph generated by ERLDAT such as the one in Figure 6, an option to save the graph as an image file is provided. The same quantile function information is provided in tabular format in Figure 7. It is important to note that this is a partial screen capture, and at the bottom of the data table are links to save the table to a pdf or Excel file.
4.2 Case Study 2

For the second case study, the approach is to design an EWMA control with \( \lambda = 0.1 \) that has a median run length of 1200. This case study is based on a similar approach taken by Gan [15]. It should also be noted that in Gan’s paper, there was only a limited number of median values, and the specific median run length of 1200 cannot be found in Gan’s paper. Furthermore, one of the major benefits of ERLDAT is that it can design for any quantile and is not limited to median run length.

To design a control with fixed \( \lambda \) to find a desired quantile performance, design case 7 is used. The use of design case 7 is illustrated in Figure [8]. After running the application, the results of searching for this control chart are shown in Figure [9]. From Figure [9], ERLDAT specifies that the value of \( k \) should be 3.237 and the ARL for this control chart is 1727.5. It is possible to verify that the median is 1200 by utilizing the Find a specific quantile feature shown in Figure [10].

The resulting output from the "Find a specific quantile" feature is illustrated in Figure [11]. The screenshot verifies that the median for this control chart is 1200. Additional information such as the probability mass function for the EWMA control chart is shown in Figure [12] and the cumulative distribution function is given in Figure [13].
Figure 8: ERLDAT Case 7 Design Page

Figure 9: ERLDAT Evaluate Page

Figure 10: ERLDAT Find Quantile Page

Figure 11: ERLDAT Quantile Results Page
Figure 12: ERLDAT PMF Graph

Figure 13: ERLDAT CDF Graph
4.3 Case Study 3

The third case study will provide another functionality not commonly available to quality practitioners; the ability to design an EWMA control chart to satisfy variance of the run length performance requirements. The Case 6 Design Page shown in Figure [14]. In this case, the values of $\lambda$, $Q_0$ and a shift size of zero is specified. ERLDAT will find the value of $k$ that has VRL of 250000. A partial screen capture of the Evaluate Page is shown in Figure [15]. The value of $k$ found by ERLDAT is 2.6266 that has an ARL of 514.41 and VRL of 250000. Note that ERLDAT users still have access to all of the evaluation functions described in the ERLDAT Evaluate Page subsection and illustrated in Figure [3].

![Figure 14: ERLDAT Case 6 Design Page](image)

![Figure 15: ERLDAT Case 6 Evaluate Page](image)

5. Conclusions and Future Research

The ERLDAT application provides quality practitioners an accurate, easy-to-use web-based application for designing and evaluation EWMA control charts. The run length performance of the ERLDAT tool was validated against published results by Crowder [1], Gan [2] and Gan [9]. The ERLDAT application calculates the run length distribution, the average run length and variance of the run length. ERLDAT design capabilities allow users to complete specify an EWMA control or search for $\lambda$ or $k$ to satisfy ARL, VRL or quantile performance requirements. Users are provided with tools to find quantiles, evaluate the pmf, cdf and survival/reliability function of the run length distribution at a given point point or generate graphs of these distribution functions. Quantiles can also be analyzed at a specified point, or tables and graphs of percentage points of the run length distributions are provided.

The authors intend to extend the ERLDAT application from designing and evaluation EWMA control charts from a standard normal distribution with $n = 1$ to any normal distribution with any subgroup size ($n$).

References


