Valuing Patents Utilizing Consensual Dynamics in a Real Options Framework

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Abstract

Patent valuation and prioritization is a regular exercise of innovative organizations. In this work, we integrate a decision support system with a unique real options framework to value and rank patents. Multiple expert opinions are used to estimate low, medium and high cash-flows for the patent projects, as well as low, medium and high patent costs. A consensual pay-off distribution is then determined for both the cash-flows and patent costs for the duration of the patent. Next, we apply a Matching Method to value the patents in a Bermudan-like real options context. We introduce two Brownian motions; each correlated to a traded index, and match one of these processes to the consensual cash-flow estimates and the other to the patent cost estimates. By applying the minimal martingale measure, the value of the cash-flows can be estimated in a risk-neutral framework. The overall methodology requires minimal subjective input of model parameters, is easy to apply in a practical setting and is consistent with financial theory.

Keywords
Real options, consensus, patents, valuation, engineering economics

1. Introduction

Analysis, selection, and valuation of patents are issues that together form a part of the work of intellectual property rights (IPR) managers, responsible for the development and management of firms’ patent portfolios. It is normal that each patent portfolio is visited according to a pre-set schedule, e.g., annually, and also when changes happen in the markets, or when new interesting patents become available that exhibit the potential to be a part of the portfolio. In this paper we discuss the issue of patent portfolio management, and more specifically, propose a new procedure that is useful in supporting patent valuation, see Figure 1. The proposed procedure is usable under conditions of parametric uncertainty; for distinctions of types of uncertainty see [1-3]. The procedure is based on using cash-flow estimates provided by multiple experts. Each expert provides cash-flow estimates for three scenarios, minimum, most likely, and maximum possible scenarios; similar to what is presented by Collan and others in [4] that is based
on the pay-off method [5,6]. The difference between the approach used here and the one used in [4,7] is that here pay-off distributions are created for yearly cash-flows from the expert-estimated scenarios and not only for the whole patent lifetime. The constructed triangular cash-flow distributions are considered to be continuous subjective probability distributions.

The theory of subjective probability, according to which probability is a theory of measurement of a person’s degree of belief in the truth of propositions [8], accommodates the classical interpretations of probability in Bayes [9] and Laplace [10] and the decision-oriented approach of Ramsey [11], de Finetti [12,13], and Savage [14]. Accordingly, subjective probabilities about some event are operationally defined as those probabilities that lead an agent to make certain choices over outcomes that depend on that event. These choices could be as natural as placing a bet on a lottery, or as structured as responding to the payoffs provided by some investment strategy evaluated using scoring rules. In order to infer subjective probabilities from observed choices of this kind, however, one either has to make some strong assumptions about risk attitudes or jointly estimate risk attitudes and subjective probabilities. If inferred subjective probabilities are conditioned on knowing risk attitudes, then any statistical uncertainty in the estimation of risk attitudes would be expected to propagate into some additional uncertainty about subjective probabilities. Usually, the estimation of subjective probabilities is done by using more than one expert because it is commonly believed that the accuracy of the final probabilities is increased by balancing multiple viewpoints and drawing from a larger pool of knowledge. The two major ways of combining the probabilities from multiple experts are aggregating individual assessments and group consensus. Since it is out of the scope of our work to enter into a description of the experimental methodologies aimed at appraising subjective probabilities, we refer the interested reader to, e.g., Andersen et al. [15].

Clearly, the cash-flows estimated by experts will likely be different from each other and consensus modeling can be used to consolidate the multiple resulting pay-off distributions into a single yearly pay-off distribution, for each patent. A graphical presentation of triangular yearly cash-flow is available, e.g., on page 7 in [16]. The consensus model that we use here was presented by Fedrizzi and others [17]. For a review of consensus modeling in the fuzzy number framework see, e.g. [18,19].

When reviewing a patent portfolio, it is possible that a patent in the portfolio has become obsolete (value has begun to deteriorate or has disappeared), or otherwise inferior to other available patents, in a way that managers wish not to include it in the portfolio anymore. This means essentially that at any moment, managers may want / are able to drop any patent out of the portfolio. This is why in a model designed to aid in patent valuation and portfolio selection, one should include a “put” option into the model, making any such model a real option valuation (ROV) model. After the triangular yearly consensus cash-flow estimates have been reached, and for accomplishing this (optionality), we turn to a “Matching Method” presented in [16] that is able to effectively map or match yearly triangular cash-flow estimates into a continuous time real option model and that allows the treatment of each underlying (in this case) patent as a real option that can be abandoned at any time. Thus, we calculate the optimal value maximizing stopping time. For further discussion on the method and how it generally fits within the real options modeling landscape we refer to [16]; more background on real option valuation modeling and the different model types can be found, e.g. from [20-24].

Figure 1. Blueprint of the proposed procedure

Collect 3 scenarios of yearly cash-flow estimates from multiple experts

Create yearly pay-off distributions from expert cash-flow estimates

Find consensus of yearly cash-flow estimates for each year

Make portfolio selection based on extended information

Run ROV with a continuous time model and find optimal value and stopping time

Convert to triangular probability density functions and match to continuous functions

Focus of this paper on the shaded area
As we assume that the expert estimated cash-flows are continuous probability distributions, they coincide with the respective probability density functions (PDF). Matching the triangular cash-flows with the matching method to continuous distributions that are “mapped” into a Markovian process, and thus, tying them into a continuous time option valuation, has the considerable benefit of linking the proposed method firmly with the classical financial option valuation theory. This allows for solving the optimal exit (abandonment) decision points of singular patents in continuous time and offers decision-makers additional decision-support in the form of revealing maximum value and when it can be extracted (according to the information available at the time of the analysis) over an “original” discrete time model. To the best of our knowledge, this is the first model combining multiple expert assessments, consensus modeling, and the matching method.

In the next section we discuss the methodology of the paper, that is the mathematical background and the models used in the proposed new procedure, then we present a numerical case study, and finally close with a short discussion and some conclusions.

2. Methodology

In this section we go through two main mathematical elements of the proposed procedure, the adopted consensus model (originally presented by Fedrizzi and others in [17]) and the new real option valuation model based on the model presented by Jaimungal and Lawryshyn in [16] and extended here.

2.1 Consensus over the decision makers pay-off distributions

We denote with \( A = \{a_1, \ldots, a_n\} \) the set of the fuzzy pay-off distribution functions (alternatives), whose estimation depends on the ideas, knowledge, attitudes, and motivations of the decision makers and, therefore, are considered as individual judgments. In consensual dynamics we introduce a mechanism driving the decision makers towards a consensus about such judgments. It is assumed that the consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator. The process starts with the representation of the fuzzy preferences of each individual decision-maker with regards to the relevant alternatives (yearly patent cash-flows). To sum up, each decision maker gives her/his opinion with respect to the yearly off distribution function estimated by any other decision maker.

Accordingly, the preferences of the decision makers are defined as subsets of \( A \times A \) and are represented with square matrices. Calling \( P^{(i)} \) the matrix representing the preference of decision maker \( i \), its element \( p_{hk}^{(i)} \) is a linguistic term denoting the degree of preference of \( a_h \) with respect to \( a_k \) as expressed by decision maker \( i \). Adopting one of the most widely used approaches to linguistically-based decision making, as introduced by L. A. Zadeh in the framework of fuzzy set theory [25,26], it is assumed that the elements are represented by a triangular fuzzy number. Starting from the results obtained in [17,27,28] we introduce a dynamic process for finding the consensual pay-off distribution reviewed, for any given patent, on the basis of a cost function \( C \) defined as a convex linear combination of a soft measure of collective dissensus \( D \) and of an inertial component of opinion changing resistance \( R \). Given the matrices \( P^{(1)}, \ldots, P^{(n)} \) of linguistically-expressed preferences of the decision makers, in order to measure their difference, one needs to calculate the distances between the triangular fuzzy numbers representing their elements. In the literature several definitions of distance between fuzzy numbers exist, see e.g. [29-32], and here a distance belonging to a family of distances introduced in [29] is adopted.

Given the two triangular fuzzy numbers \( x = (\delta_L, x, \delta_R) \) and \( y = (\varepsilon_L, y, \varepsilon_R) \), where \( x \) and \( y \) are the central values and \( \delta_L, \delta_R, \varepsilon_L, \varepsilon_R \) are the left and right spreads respectively. The distance is calculated as

\[
d(x, y) = \frac{d_L + d_R}{2}
\]

where \( d_L \) and \( d_R \) are computed using integrals

\[
d_L = \int_0^1 (x_L(\alpha) - y_L(\alpha))^2 \, d\alpha, \quad d_R = \int_0^1 (x_R(\alpha) - y_R(\alpha))^2 \, d\alpha
\]

where the difference is calculated using left and right side \( \alpha \)-cuts of the triangular fuzzy numbers which are given as

\[
[x_L(\alpha), x_R(\alpha)] = [-\delta_L + \delta_R \alpha, x + \delta_R \alpha], \quad [y_L(\alpha), y_R(\alpha)] = [-\varepsilon_L + \varepsilon_R \alpha, y + \varepsilon_R \alpha]
\]
for each \( \alpha \in [0, 1) \).

This distance is obtained from the family of distances defined in [29]. Let us remark that \( d \) is not exactly a distance because it does not always satisfy the transitivity axiom, nevertheless for the sake of simplicity, the term distance is used when referring to \( d \). By solving the integrals \( d_L \) and \( d_R \) we get

\[
d(x, y) = \Delta_D^2 + \Delta_L^2/6 + \Delta_R^2/6 + \Delta_D(\Delta_R-\Delta_L)/2
\]

where \( \Delta_D = x-y, \Delta_L = \delta_L - \varepsilon_L, \) and \( \Delta_R = \delta_R - \varepsilon_R \).

Assuming now, for the sake of simplicity, that only two alternatives are involved, we indicate with \( p^{(i)}= (\delta_l^{(i)}, p^{(i)}, \delta_r^{(i)}) \) and \( p^{(j)}= (\varepsilon_l^{(j)}, p^{(j)}, \varepsilon_r^{(j)}) \) the preferences expressed by the decision makers \( i \) and \( j \) respectively. Following the dynamic consensus process introduced in [27], we determine the global dissensus measure of the group of decision makers

\[
D = \frac{1}{4} \sum_{i=1}^{n} C_1(i),
\]

where \( C_1(i) \) is given as

\[
C_1(i) = \frac{\left( \sum_{j \neq i} C_1(i, j) \right)}{(n-1)}
\]

and

\[
C_1(i, j) = f(d(p^{(i)}, p^{(j)})).
\]

Here \( f(\cdot) \) is a scaling function defined as \( f(t) = \frac{1}{\beta} \ln(1+e^{\beta(x-\alpha)}) \), where \( \alpha \in (0, 1) \) is a threshold parameter and \( \beta \in (0, \infty) \) is a free parameter controlling the polarization of the sigmoid function \( f'(x) = 1/(1+e^{\beta(x-\alpha)}) \). The cost function for changing the initial preference \( \pi^{(i)} \) of decision maker \( i \) into the new preference \( p^{(i)} \) is given as

\[
C_2(i) = f(d(p^{(i)}, \pi^{(i)})).
\]

Accordingly, the global opinion changing aversion of the group of decision makers is given by summing up the individual one to \( C_2 \) by

\[
R = \frac{1}{2} \sum_{i=1}^{n} C_2(i).
\]

The global cost function \( C \) can now be given as

\[
C = (1-\lambda) D + \lambda R
\]

where \( \lambda \in [0, 1] \) is a parameter representing the relative importance of the inertial component \( R \) w.r.t. the dissensus component \( D \). The consensual dynamics is based on the minimization of the cost function \( C(p^{(i)})=C(\delta_l^{(i)}, p^{(i)}, \delta_r^{(i)}) \) through the standard gradient method. The new preference, for any decision maker, is obtained from the previous one according to the following iterative process

\[
p \rightarrow p^* = p - \gamma \nabla C.
\]

The components of the gradient \( \nabla C \) are obtained deriving \( C \) with respect to \( \delta_l^{(i)}, p^{(i)}, \) and \( \delta_r^{(i)} \). For closer formulation of the iterative process we refer to Fedrizzi and others in [17]. Now, using the above described method for reaching consensus we can create, for each patent, a triangular cash-flow distribution for each year of the estimated economic life of the patent (from the estimations of multiple experts for the same).
2.2. The proposed real option valuation model

As discussed above, each patent valuation will have associated with it a consensus determined triangular cash-flow probability density function for each year of the estimated duration of the life of the product (patent). We further assume that the patent investment costs, which include patent maintenance costs, renewal fees and potential litigation costs, need to be paid on a periodic basis and these costs, too, are estimated as low, medium and high, which we also assume to be triangularly distributed. While it is likely that the cash-flows have a strong systematic component, it is likely that the patent costs are mostly idiosyncratic to the market. The ability to account for market correlation of the cash-flows and patent costs separately, is one of the key features of our model. We further assume that managers have the option to not invest (further) in the patent, if the valuation is negative, in which case they forego any future cash-flows. Clearly, management would typically choose not to invest under scenarios where the future cash-flows are not sufficient enough to cover the cost of the patent investment. We assume that these decisions are made at each patent cost investment date, which leads to a series of call options.

We assume there exists a traded index \( I_t \) that follows geometric Brownian motion (GBM),

\[
\frac{dl_t}{l_t} = \mu dt + \sigma dB_t,
\]

where \( B_t \) is a standard Brownian motion under the real-world measure \( \mathbb{P} \), and \( \mu \) and \( \sigma \) are the drift and volatility parameters, assumed to be constant. As in [16], we introduce two observable but not tradable processes ([16] used a GBM but here Brownian Motion is used; it can be shown that using a Brownian motion leads to an identical valuation under the European option assumption),

\[
dX_t = \rho_X dB_t + \sqrt{1 - \rho^2_X} dW^X_t \tag{14}
\]

and

\[
dY_t = \rho_Y dB_t + \sqrt{1 - \rho^2_Y} dW^Y_t, \tag{15}
\]

where \( W^X_t \) and \( W^Y_t \) are two other standard Brownian motions under the real world measure \( \mathbb{P} \), independent from each other and \( B_t \), and \( \rho_X \) and \( \rho_Y \) are constant correlations. We also assume that \( X_0 = 0 \) and \( Y_0 = 0 \). As discussed in [16], \( X_t \) and \( Y_t \) can be considered as appropriate Sector Indicators, and we will use \( X_t \) to drive the consensus cash-flow estimates and \( Y_t \) to drive the patent cost estimates.

We assume that the cash-flows \( S_k \) occur at times \( T_k \) \((k = 1, 2, ..., n)\) and the patent investment costs \( M_k \) occur at \( T'_k \), i.e. the cash-flow for the \( k \)-th time period (typically a year) is received before the next investment in the patent is required. This is a standard assumption as it is clear that the cash-flows will be generated on a daily basis throughout each time period, whereas the patent investment costs will only be required on a periodic, likely annual, basis. We define a collection of functions \( \phi^S_k \) and \( \phi^M_k \) such that

\[
S_k = \phi^S_k \left( X_{T_k} \right) \quad \text{and} \quad M_k = \phi^M_k \left( Y_{T'_k} \right), \tag{16}
\]

and require that at all dates, \( T_k \), the managerial triangular distribution estimates are matched. Thus, we require that

\[
\mathbb{P}(S_{T_k} < s) = F^S_k(s) \quad \text{and} \quad \mathbb{P}(M_{T'_k} < m) = G^M_k(m), \tag{17}
\]

where \( F^S_k(s) \) and \( G^M_k(s) \) are the managerial supplied triangular distributions for the \( k \)-th cash-flow and patent cost, respectively. Clearly, there will be no patent cost at \( k = n \) and there may not be any cash-flows expected for lower values of \( k \). In such cases we set \( \phi^S_k(\cdot) = 0 \) or \( \phi^M_k(\cdot) = 0 \), as appropriate.

We note that for a given triangular density with a minimum value of \( y_- \), a peak at \( y_0 \) and a maximum value at \( y_+ \), we have
Y. Lawryshyn, M. Collan, P. Luukka and M. Fedrizzi.

\[ F(y) = \begin{cases} 
0, & y \leq y_-, \\
\frac{a (y - y_-)^2}{2 y_0 - y_-}, & y_- < y \leq y_0, \\
\frac{1 - a (y_+ - y)^2}{2 y_0 - y_+}, & y_0 < y \leq y_+, \\
1, & y > y_+, 
\end{cases} \]  \tag{18}

where \( a = \frac{2}{(y_+ - y_-)}. \) The inverse of \( F \) is given by

\[ F^{-1}(p) = \begin{cases} 
0 + \sqrt{(y_+ - y_-)(y_0 - y_-)p}, & p \leq p_c, \\
0 + \sqrt{(y_+ - y_-)(y_0 - y_0)(1 - p)}, & p > p_c, 
\end{cases} \]  \tag{19}

where the critical point \( p_c = \frac{(y_0 - y_-)}{(y_+ - y_-)}. \)

**Proposition 1 Replicating the Cash-Flows and Patent Costs.** The cash-flow matching function \( \varphi_k^S(x) \) which produces the managerial specified distribution \( F_k^*(s) \) for the cash-flows at time \( T_k \), when the underlying driving uncertainty \( X_t \) is a Brownian motion, is given by

\[ \varphi_k^S(x) = F_k^{*^{-1}} \left( \Phi \left( \frac{x}{\sqrt{T_k}} \right) \right) \]  \tag{20}

where \( \Phi(*) \) is the standard normal distribution. Analogously, for the patent costs we have

\[ \varphi_k^M(y) = G_k^{*^{-1}} \left( \Phi \left( \frac{y}{\sqrt{T_k}} \right) \right). \]  \tag{21}

**Proof:** We seek \( \phi(*) \) such that \( \mathbb{P}(\phi(X_T) \leq s) = F^*(s) \). Since \( X_T \sim \mathcal{N}(0,1) \), we have

\[ \mathbb{P}(\phi(X_T) \leq s) = \mathbb{P}(\sqrt{T}Z \leq \phi^{-1}(s)) = F \left( \phi^{-1}(s) \right) \]  \tag{22}

Thus, if \( F^* \) is invertible then

\[ \phi(X_T) = F^{-1} \left( \Phi \left( \frac{X_T}{\sqrt{T}} \right) \right). \]  \tag{23}

as required.

At each \( T_k \) we have the cash-flow payoff and associated patent cost. We define the total cash flow as

\[ v_k = \varphi_k^S - \varphi_k^M. \]  \tag{24}

To be consistent with economic theory, the value of the cash-flows and patent costs should be determined under a risk-neutral measure. However, as neither \( X_t \) nor \( Y_t \) are assumed to be traded assets, the market in incomplete. An appropriate risk-neutral measure \( \mathbb{Q} \), corresponding to a variance minimizing hedge, is one where we have the following dynamics

\[
\begin{align*}
\text{d}I_t &= rI_t \text{d}t + \sigma_I \text{d}B^I_t, \\
\text{d}X_t &= \hat{v} \text{d}t + \rho_X \text{d}B^I_t + \sqrt{1 - \rho_X^2} \text{d}W^X_t, \\
\text{d}Y_t &= \hat{v} \text{d}t + \rho_Y \text{d}B^I_t + \sqrt{1 - \rho_Y^2} \text{d}W^Y_t
\end{align*}
\]  \tag{25-27}

where \( r \) is the risk-free rate and \( \hat{B}^I_t, \hat{W}^X_t \) and \( \hat{W}^Y_t \) are standard uncorrelated Brownian motions under the risk-neutral measure \( \mathbb{Q} \) with the risk-neutral drifts given by
\[
\dot{\nu} = -\rho X \frac{\mu - \nu}{\sigma} \quad \text{and} \quad \dot{\gamma} = -\rho Y \frac{\mu - \nu}{\sigma}.
\] (28)

At \( t > T_{n-1}^+ \), the value of the cash-flow is given by

\[
V_n^+(X_t) = e^{-r(T_t - 1)}E^Q \left[ \varphi_n^+(X_T) \mid X_t \right].
\] (29)

At \( t = T_{n-1}^+ \) the managers have the option to decide if they should invest the (last) patent fee payment. Thus at

\[
V_n^{n-1}(X_{T_{n-1}}, Y_{T_{n-1}}) = e^{-r(T_{n-1} - 1)}E^Q \left[ \left( V_n^{n-1}(X_{T_{n-1}}) - \varphi_M^+(Y_{T_{n-1}}) \right) + \varphi_n^+(X_T) \mid X_{T_{n-1}}, Y_{T_{n-1}} \right].
\] (30)

Similarly, at \( T_{k-1}^+ < t \leq T_k \), for \( k = 1, 2, \ldots, n - 2 \), with \( T_0^+ = 0 \),

\[
V_k^+(X_t, Y_t) = e^{-r(T_{k-1} - 1)}E^Q \left[ \left( V_k^{k+1}(X_{T_{k-1}}, Y_{T_{k-1}}) - \varphi_M^+(Y_{T_{k-1}}) \right) + \varphi_k^+(X_T) \mid X_{T_{k-1}}, Y_{T_{k-1}} \right].
\] (31)

Given that the value of the total cash-flow is a function of the two processes, \( X_t \) and \( Y_t \), applying Ito’s lemma, the value function \( V_t(X, Y) \) can be determined by solving the following PDE

\[
rV = \frac{\partial V}{\partial t} + \dot{\nu} \frac{\partial V}{\partial \nu} + \dot{\gamma} \frac{\partial V}{\partial \gamma} + \frac{1}{2} \frac{\partial^2 V}{\partial \nu^2} + \frac{1}{2} \frac{\partial^2 V}{\partial \gamma^2} + \rho_X \rho_Y \frac{\partial^2 V}{\partial \nu \partial \gamma}
\] (32)

with the terminal boundary condition given as \( V(X_{T_k}, Y_{T_k}) = V_k^{T_k}(X_{T_k}) \), and, at each \( T_k, k = 1, 2, \ldots, n - 1 \) we have \( V(X_{T_k}, Y_{T_k}, T_k) = V_k^{T_k}(X_{T_k}, Y_{T_k}) \). At the spatial boundaries, we assume zero curvature.

3. Case study

In this section we present a practical case study, where we illustrate how the optimal stopping (abandonment, exit) point for each patent can be calculated. The company under consideration is looking to invest in 20 new patents. Cash-flow estimates for each patent were developed by four independent decision makers. In each case, the patents are expected to expire after 9 years, after which we have assumed there would be no future cash flows. The patent investment cost estimates are provided in Appendix B (available on request) for years 0 to 2, as these estimates vary somewhat, depending on the project associated with the patent.

These estimates were assumed to be presented by a single subject matter expert and therefore no consensus calculations were required. The patent maintenance costs for years 3 to 8 were assumed to be the same for all projects and are presented in an Appendix C (available on request). The low cost estimate was set at €3000, the medium at €3300, growing at 2% per year, and the high at €4000, growing at 5%. Consensus calculations for all cash-flow estimates were performed and are presented in Appendix A and are available on request. In all calculations we assumed that \( \alpha = 0.3 \) and \( \beta = 10 \) for threshold and polarization control parameter effecting the iteration speed and tested to be suitable for particular problem. A total of 5000 iterations were performed to make sure the results converged. We note that with this choice of parameters, the consensus low, medium and high values are very close to the average values provided by the decision makers.
Y. Lawryshyn, M. Collan, P. Luukka and M. Fedrizzi.

Table 1. Patent project values for varying cash-flow to market correlation, $\rho_X$ (Euros).

<table>
<thead>
<tr>
<th>Patent No.</th>
<th>Patent Value for Given $\rho_X$</th>
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<tr>
<td></td>
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<tr>
<td>1</td>
<td>49,679</td>
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<tr>
<td>2</td>
<td>246,853</td>
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<td>3</td>
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<td>19</td>
<td>101,120</td>
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<tr>
<td>20</td>
<td>427,963</td>
</tr>
</tbody>
</table>

As discussed in section 2.2., our model properly accounts for systematic versus idiosyncratic risk. While it is likely that the patent costs will not have a strong correlation to the market, and therefore we appropriately assumed $\rho_Y = 0.1$, it is likely that the cash-flows will be influenced by the market, thus the valuations were carried out for varying $\rho_X$ values and are presented in Table 1. The valuations were performed with market parameters: $r = 3\%$, $\mu = 0.9$ and $\sigma = 0.1$. As expected, the greater the correlation, the lower the valuation. This result is financially consistent for the greater the systematic risk, the greater the project beta and thus, the greater the appropriate discount factor.

Figure 2. Patent project number 8 value as a function of processes $X_t$ and $Y_t$. 
The patent project value for patent 8 is plotted as a function of \(X_t\) and \(Y_t\) at time 0 for the case of \(\rho = 0.5\) in Figure 2. The value of the function at point \(X_0 = 0, Y_0 = 0\) is the represents the value of this project. The corresponding exercise boundary for (optimally, with highest retained value) quitting the project is presented in Figure 3. As can be seen, for this particular project, there is a chance that it would be advantageous to quit this project at or before year 8, depending on the evolution of the processes \(X_t\) and \(Y_t\).

![Figure 3. Exercise boundary for patent project number 8 as a function of processes \(X_t\) and \(Y_t\).](image)

Note that \(X_t\) and \(Y_t\) originate at \((0,0)\) at \(t = 0\) and evolve accordingly. If, at any time \(t \leq 7\), the values enter the region marked \(t = 0\), then it is optimal to abandon, similarly for \(t \leq 8\), it is optimal to abandon if the region marked \(t = 7\) is entered (note that one would always enter the \(t = 7\) region before entering \(t = 0\)), and it would be optimal to abandon for \(t \geq 8\) if the \(t = 8\) region is entered.

We can now calculate the optimal stopping (abandonment) point with the optimal value for each patent, based on the PDFs and use this information in the selection and portfolio optimization of our patent portfolio. This is additional information to the information provided by using the original pay-off method alone and allows the decision-makers to extract the most value from the patents (by being able to discontinue or otherwise exercise them optimally).

4. Discussion and conclusions

We have presented a new approach for the valuation of patents as real options in continuous time, based on consensus estimates of triangular yearly cash-flow estimates. The method is a good representation of reality, because it treats each patent as a real option that can be abandoned at any time (optimal time) and offers decision-makers additional information for patent portfolio-optimization over previous methods using similar data. The method is easy to use, for it requires minimal subjective input regarding model parameters; for example, there is no requirement for the calculation of a discount rate – the riskiness of the cash-flows are inherently captured by the uncertainty of their respective distributions. Furthermore, it is generally accepted that experts prefer to provide a range of possibilities, rather than a single value and the proposed method clearly takes advantage of this for the main inputs are ranges of cash-flows and costs. As well, the method is firmly linked with classical financial theory, and, specifically, is able to properly account for systematic versus idiosyncratic risk. Finally, we have illustrated, with a numerical example, how the method can provide optimal “stopping times” for holding patents.

We have prepared Appendixes for yearly patent cash-flow estimates and for their consensus results and for patent cost estimates; these are available at request.
**References**

23. Trigeorgis L., Real Options: An Overview, Preager, Westport, CT.